MASS-WEIGHTED SYMPLECTIC FORMS FOR THE N-BODY PROBLEM

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Abstract. Mass-weighted symplectic forms provide a unified framework for the treatment of both finite and vanishingly small masses in the N-body problem. These forms are introduced, compared to previous approaches, and their properties are discussed. Applications to symplectic mappings, the definition of action-angle variables for the Kepler problem, and Hamiltonian perturbation theory are outlined.

Key words: symplectic form, symplectic mapping, Hamiltonian formalism, perturbation theory, N-body problem.

1. Introduction

The Hamiltonian formalism and the associated symplectic structures provide the basis for many of the latest developments in Solar System dynamics. Symplectic mapping techniques have become the method of choice for very long numerical integrations of planetary orbits (Sussman and Wisdom, 1992). Among these, Wisdom and Holman's (1991, 1992) technique utilizes the fact that the motion of each planet is dominated by the integrable Kepler problem. An essential ingredient in their method is the use of Jacobi coordinates, which make it possible to split each planet's Hamiltonian into a Keplerian and a perturbation part. Analytic and semi-analytic computations, such as perturbation theories, often rely on the Hamiltonian formalism. Hamiltonian perturbation theory is also employed to analyze various properties of symplectic mappings (Wisdom and Holman, 1992; Saha and Tremaine, 1992; Yoshida, 1993; Mikkola, 1997). In all cases, the underlying phase-space structure is defined by the symplectic form and the associated Poisson brackets.

The role that the masses play in these computations is somewhat obscure. If one starts with finite masses, it soon becomes hard to interpret the formulae for vanishingly small ones, *i.e.*, for test particles such as the asteroids, especially in the presence of nongravitational forces (Wolansky *et al.*, 1998). The usual approach to deal with such cases is to introduce time-dependent Hamiltonians that are forced by the motion of the particles having finite masses. There appears to be no smooth transition from finite masses to vanishingly small ones. This lack of continuity does not seem to be related to any actual physical phenomenon, as the original Newtonian equations of motion do not break down when masses become vanishingly small. One might suspect that the symplectic structure, at least in the form generally used, is responsible for these complications. The difficulties are created, as it turns out, by adopting linear momenta as dynamical variables.

In this paper we describe a simple way to treat masses in a uniform manner, *i.e.*, to derive formulae which are valid for both actual and test particles. This is achieved by factoring out the masses from the respective terms in the symplectic form. The resulting mass-weighted symplectic forms (MWSFs) offer simplifications that have been mostly overlooked so far. The case of vanishingly small masses was studied by Varadi *et al.* (1995a) but it was limited to special types of linear transformations and Lie-series computations. Here we consider the general case. The results are fairly simple to derive and the emphasis is on exploring connections between various structures.

In Section 2 the application of MWSFs to the definition of action-angle variables for the Kepler problem is described. In Section 3 we formulate Jacobi coordinates through noncanonical transformations and offer insight into their connection with MWSFs. It appears to be more natural to use a simple transformation to Jacobi coordinates than to insist on the exact preservation of the symplectic form. In Section 4, practical issues regarding the implementation of symplectic mappings for so-called hierarchical dynamical systems (Roy, 1988) are discussed. The application of MWSFs to Hamiltonian perturbation theory is outlined in Section 5, and concluding remarks follow in Section 6.

2. The basic properties of MWSFs

In order to motivate introducing MWSFs, one could repeat the arguments put forward in favor of singularly weighted ones (Varadi *et al.*, 1995a). It is more useful, however, to consider one particular aspect of the problem, namely, the definition of action-angle variables for Kepler motion.

Let $q_i, i = 1, 2, 3,$ denote the Cartesian coordinates of a particle with mass m and let μ denote the product of the central mass and the constant of gravity. The standard symplectic form is then (Arnold, 1988)

$$\omega = \mathrm{d}p_1 \wedge \mathrm{d}q_1 + \mathrm{d}p_2 \wedge \mathrm{d}q_2 + \mathrm{d}p_3 \wedge \mathrm{d}q_3, \tag{1}$$

where the p_i's denote the components of the linear momentum, *i.e.*, $p_i = m\dot{q}_i$. The symplectic form ω acts on pairs of vector fields — such as the differential equations for the positions (q_i) and momenta (p_i) — in a bilinear and skew-symmetric fashion and thus can be represented by a matrix (Abraham and Marsden, 1978; Arnold, 1988). The inverse of this matrix maps differentials of functions into vector fields and thus makes it possible to represent systems of differential equations by functions, *i.e.*, to obtain equations of motions from Hamiltonians. The Poisson bracket of two functions is the symplectic form's value on the vector fields associated with the given functions; the bracket is thus again a function. In the case of Eq. (1), the matrix of ω has a simple bidiagonal structure and the general mathematical constructions boil down to the Hamiltonian formalism's familiar rules.

One can write the Hamiltonian for Kepler motion (e.g., Message, 1995) as

$$H = \frac{p^2}{2m} - \frac{\mu m}{\|q\|} = \frac{1}{m}\left(\frac{p^2}{2} - \frac{\mu m^2}{\|q\|}\right), \tag{2}$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$ and $\|q\| = \sqrt{q_1^2 + q_2^2 + q_3^2}$. Assuming that $m \neq 0$, one can carry out well-known transformations to action-angle variables, *e.g.*, to Delaunay variables (Arnold, 1988; Message, 1995); doing so yields

$$H = -\frac{1}{m}\frac{\mu^2 m^4}{2L^2}, \tag{3}$$

where $L = m\sqrt{\mu a}$ is an action variable and a is the semi-major axis. When $m = 0$, the components of the linear momentum p_i vanish, along with L and H, but the velocities $\dot{q}_i$ do not.

The fact that L is proportional to the mass of the particle introduces complications into any Hamiltonian perturbation computation. The expansion of the perturbing function in canonical variables becomes intractable as masses appear in the expression of distances between the planets. The practical usefulness of expanding the perturbing function in mass-dependent canonical variables is questionable. In symbolic algebraic manipulations, which may involve repeated Poisson brackets of hundreds of thousands of terms (Varadi *et al.*, 1995b; Ghil *et al.*, 1996) the computations have to be automated. The more complicated the dynamical variables, the more error-prone the computer codes become. Furthermore, one needs to be able to evaluate the relative importance of terms in these expansions and how they change as functions of the parameters, such as the masses. The original motivation for considering MWSFs was to eliminate the masses from the canonical variables as much as possible.

The equations of motion for a test particle of zero mass can be derived from a time-dependent Hamiltonian. These equations are independent of the particle's mass and one can thus take it to be unity. This apparent contradiction of having zero and unit mass at the same time is never taken as a serious impediment in practice but it still seems worthwhile to find a simple resolution to it.

In order to obtain a uniformly valid formalism for both finite and vanishingly small masses, one can rewrite the symplectic form in terms of velocities and coordinates. The standard symplectic form (1) becomes a mass-weighted one, *i.e.*,

$$\omega = m\left(\mathrm{d}v_1 \wedge \mathrm{d}q_1 + \mathrm{d}v_2 \wedge \mathrm{d}q_2 + \mathrm{d}v_3 \wedge \mathrm{d}q_3\right); \tag{4}$$

we thus have

$$H = \frac{mv^2}{2} - \frac{\mu m}{\|q\|} = m\left(\frac{v^2}{2} - \frac{\mu}{\|q\|}\right), \tag{5}$$

where $v = \dot{q}$ denotes velocity. Having the mass as a multiplier in the symplectic form introduces the same mass as a divisor in the Poisson bracket (Varadi *et al.*, 1995a) since the latter involves inverting the matrix of the former. Thus we have

$$\{q_i, v_i\} = \frac{1}{m} \tag{6}$$

and all other types of brackets are zero.

The mass now appears as a multiplier in the Hamiltonian and as a divisor in the Poisson bracket, and the two cancel each other in the equations of motion. This happens irrespectively of the actual mass, vanishingly small or otherwise. Having observed this, the rest is straightforward computation. For instance, in terms of action-angle variables one obtains

$$H = -\frac{m\mu^2}{2L^2}, \tag{7}$$

where $L = \sqrt{\mu a}$. The scaling of L by the mass of the particle is eliminated and the case $m = 0$ no longer requires any special treatment.

Weighted symplectic forms are not new. A similar construction is used in the dynamics of point vortices, where the intensity of the vortices replace the masses (Arnold, 1988). In the spirit of full mathematical rigor, the v_i's should not be called velocity components since the Legendre transformation between velocities and momenta (Abraham and Marsden, 1978; Arnold, 1988) still holds. In other words, the vector (v_1, v_2, v_3) is an element in the cotangent bundle of the underlying con£guration manifold and not in its tangent bundle, where the usual velocity vector belongs. The term co-velocity would be more precise but for our purposes it is sufficient to apply the traditional identi£cation of a vector space with its dual space. For the sake of simplicity, we thus use the term velocity, while keeping in mind the £ner point involved.

For a system of N particles with masses m_i, positions q_i and velocities v_i, $i = 1, 2, \ldots N$, one can use the symplectic form

$$\omega = \sum_{i=1}^{N} m_i\left(\mathrm{d}v_{i,1} \wedge \mathrm{d}q_{i,1} + \mathrm{d}v_{i,2} \wedge \mathrm{d}q_{i,2} + \mathrm{d}v_{i,3} \wedge \mathrm{d}q_{i,3}\right). \tag{8}$$

The corresponding Poisson bracket is

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{m_i} \sum_{j=1}^{3} \frac{\partial f}{\partial q_{i,j}} \frac{\partial g}{\partial v_{i,j}} - \frac{\partial g}{\partial q_{i,j}} \frac{\partial f}{\partial v_{i,j}}, \tag{9}$$

where the masses appear as divisors. If the interaction between the particles is such that the potential energy for particles i and j is proportional to the product of their masses $m_i m_j$, then the masses are selective divisors. More exactly, they divide expressions which already have the same masses as multipliers.

This is true for the gravitational N-body problem with the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{m_i}{2} v_i^2 - \sum_{i<j} \frac{G m_i m_j}{r_{ij}}, \tag{10}$$

where $v_i^2 = \sqrt{v_{i,1}^2 + v_{i,2}^2 + v_{i,3}^2}$, r_{ij} denotes the distance between particle i and j, and G is the constant of gravity. One can observe that what really affects particle i is only

$$H_i = \frac{m_i}{2} v_i^2 - \sum_{j} \frac{G m_i m_j}{r_{ij}}, \tag{11}$$

which is proportional to m_i. Thus the masses in the Poisson bracket selectively cancel the masses in the Hamiltonian.

3. Application to Jacobi coordinates

Since MWSFs appear to be quite natural for the gravitational N-body problem, we next discuss their use in formulating symplectic mappings. The Wisdom-Holman mapping achieves the splitting of the Hamiltonian into Keplerian and perturbation parts by introducing Jacobi coordinates. Usually, the symplectic character of the Wisdom-Holman mapping after the transformation to Jacobi coordinates is emphasized over that of the transformation itself. Wisdom and Holman (1991) derive the new momenta from the time derivative of the Jacobian positions and it is not immediately clear that the resulting transformation is canonical. In this section we demonstrate that from the point of view of MWSFs the classical, noncanonical formulation of Jacobi coordinates is more natural than a canonical one.

Let us consider two particles with masses m_1 and m_2. For the sake of brevity, we consider only one spatial coordinate q and the corresponding velocity v. The mass-weighted symplectic form for the two particles is

$$\omega = m_1 \mathrm{d}v_1 \wedge \mathrm{d}q_1 + m_2 \mathrm{d}v_2 \wedge \mathrm{d}q_2. \tag{12}$$

We also need the canonical 1-form (Abraham and Marsden, 1978)

$$\theta = m_1 v_1 \mathrm{d}q_1 + m_2 v_2 \mathrm{d}q_2, \tag{13}$$

which is related to the symplectic form through exterior differentiation, namely,

$$\omega = \mathrm{d}\theta. \tag{14}$$

If a transformation preserves the canonical 1-form θ, then it also preserves the symplectic form ω, *i.e.*, the transformation is canonical, since the exterior derivative operator d is natural with respect to diffeomorphisms (Abraham and Marsden, 1978).

The Jacobi coordinates are defined as

$$Q_1 = \frac{m_1}{m_1+m_2} q_1 + \frac{m_2}{m_1+m_2} q_2, \tag{15}$$

$$Q_2 = q_2 - q_1, \tag{16}$$

where Q_1 is the barycenter and Q_2 is the relative position of the two particles. The transformation of the positions can be extended to the whole phase space (Q_1, Q_2, V_1, V_2) of positions and velocities to obtain a canonical transformation by the formulae

$$v_1 = \frac{m_1}{m_1+m_2} V_1 - \frac{m_2}{m_1} V_2, \tag{17}$$

$$v_2 = \frac{m_1}{m_1+m_2} V_1 + V_2, \tag{18}$$

where the original velocities are expressed in terms of the new ones. This is the usual lift of the transformation of the positions (Abraham and Marsden, 1978), modified for the case of the MWSF (12) at hand; thus θ and ω are both preserved. The total kinetic energy

$$K = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) \tag{19}$$

becomes

$$K = \frac{1}{2}\left(\frac{m_1^2}{m_1+m_2} V_1^2 + \frac{m_2(m_1+m_2)}{m_1} V_2^2\right). \tag{20}$$

One can compare Eq. (20) to the 1-form

$$\theta = m_1 V_1 \mathrm{d}Q_1 + m_2 V_2 \mathrm{d}Q_2 \tag{21}$$

and observe that the identi£cation of masses is not clear anymore: there are now two different mass-like quantities, the ones which are multipliers in θ and the ones which are multipliers in the total kinetic energy K. Since Q_1 represents the barycenter of the two particles, one might expect that at least one of the two kinds of multipliers, the one associated with Q_1 in Eq. (21) would be the total mass $m_1 + m_2$. But this is not the case and there appears to be no obvious remedy to the confusion so created.

On the other hand, using the same transformation for the coordinates and velocities, *i.e.*, the classical form of Jacobi coordinates, gives the formulae

$$v_1 = V_1 - \frac{m_2}{m_1+m_2} V_2, \tag{22}$$

$$v_2 = V_1 + \frac{m_1}{m_1 + m_2} V_2. \tag{23}$$

Since this transformation is not canonical, one has to express the 1-form θ of Eq. (13) in terms of the new variables. The result is

$$\theta = (m_1 + m_2) V_1 \mathrm{d}Q_1 + m_2 \left(1 - \frac{m_2}{m_1 + m_2}\right) V_2 \mathrm{d}Q_2. \tag{24}$$

The total kinetic energy K can be determined from the last formula without additional algebra by observing that the same linear transformation applies to both positions and velocities. The expressions $\mathrm{d}Q_i$ in θ can be replaced by the corresponding V_i's to obtain the formula for K, *i.e.*,

$$K = \frac{1}{2} \left[(m_1 + m_2) V_1^2 + m_2 \left(1 - \frac{m_2}{m_1 + m_2}\right) V_2^2 \right]. \tag{25}$$

In this case we have the same mass-like quantities in the total kinetic energy (25) and in the 1-form θ of (24). The new mass associated with the barycenter is the total mass, while the relative coordinates are associated with a reduced mass. The reduced mass can also be written as

$$m_2 \left(1 - \frac{m_2}{m_1 + m_2}\right) = \frac{m_1 m_2}{m_1 + m_2}; \tag{26}$$

its effect through the Poisson bracket is to turn the product $m_1 m_2$ in the Hamiltonian into the sum $m_1 + m_2$ in the equations of motion.

For a system of particles one can use the same construction recursively by forming the pairwise union of subsystems. At each step one introduces the barycenter of the union of the two subsystems and the relative position of the two barycenters. The former is associated with the total mass of the two subsystems, while the latter is associated with a reduced mass.

Wisdom and Holman's (1991) canonical derivation of Jacobi coordinates has a simple interpretation. One can re-introduce the linear momenta into the formulae of this section and appropriately scale the new momenta to obtain a canonical transformation. This would re-introduce the difficulties with vanishingly small masses and it seems better to keep the 1-form θ and the corresponding symplectic form $\omega = \mathrm{d}\theta$ in their mass-weighted forms, cf. Eq. (24).

The fact that both positions and velocities are transformed by the same formula also gives the expression for the total angular momentum in terms of Jacobi coordinates. First, since the exterior derivative operator d is natural with respect to diffeomorphisms (Abraham and Marsden, 1978), the relationship (14) holds in the Jacobi coordinates (Q_1, Q_2, V_1, V_2), *i.e.*,

$$\omega = (m_1 + m_2)\, \mathrm{d}V_1 \wedge \mathrm{d}Q_1 + \frac{m_1 m_2}{m_1 + m_2}\, \mathrm{d}V_2 \wedge \mathrm{d}Q_2. \tag{27}$$

Second, ω behaves the same way with respect to linear transformations as the total angular momentum, since both are defined by skew-symmetric bilinear forms and the exterior derivative d is a linear operator. Note that the skew-symmetric bilinearity of the vector product is the reason why it is sometimes denoted by $\wedge$. Explicitly, the total angular momentum

$$\mathbf{M} = m_1\mathbf{q}_1 \times \mathbf{v}_1 + m_2\mathbf{q}_2 \times \mathbf{v}_2 \tag{28}$$

becomes

$$\mathbf{M} = (m_1 + m_2)\,\mathbf{Q}_1 \times \mathbf{V}_1 + \frac{m_1 m_2}{m_1 + m_2}\,\mathbf{Q}_2 \times \mathbf{V}_2, \tag{29}$$

where all positions and velocities are now three-dimensional vectors. Thus the barycentric term is multiplied by the total mass, and the perturbation term by the reduced mass.

4. Application to hierarchical N-body systems

A description of Jacobi coordinates based on Newton's equations of motion is provided by Roy (1988) for the general case of hierarchical dynamical systems. The formulae are applicable to a wide range of N-body problems, from the Solar System to multiple-star systems. Such dynamical con£gurations are best described by a tree of barycenters. In the case of two particles, one introduces their barycenter and relative position in place of their original positions in an inertial frame, as we did in the previous section. In the general case, one can arrange the particles into a binary tree. Two subsystems are combined by creating a new branch point of the tree, to which the two subsystems are attached, and the new branch point represents the barycenter of the combined system. This new barycenter and the relative position of the barycenters of the two subsystems give the Jacobi coordinates which replace the original positions of the two barycenters. The velocities are transformed by the same formulae as the positions.

The derivation of symplectic mappings for a hierarchical N-body problem follows from the recursive de£nition of Jacobi coordinates. The Hamiltonian can be expressed in terms of Jacobi coordinates but, as it turns out, its explicit form is not necessary. One only needs to keep track of the barycenters of the subsystems, *i.e.*, the binary tree representing the hierarchy. Let us consider two subsystems of particles, A and B, with total masses M_A and M_B, and let x_A and x_B denote the position vectors of their barycenters. At this stage one only has to take care of the interaction between the two subsystems; whatever needs to be done inside each subsystem will follow from a recursive application of the same formulae.

The interaction of A and B is supposed to be dominated by the Keplerian motion of their relative barycenters. When the two subsystems are combined, one can introduce the term

$$U_{\mathrm{K}} = -\frac{G M_A M_B}{\|x_A - x_B\|} \tag{30}$$

into the Hamiltonian as the Keplerian part of the interaction between A and B. The behavior of the relative position of the barycenters $x_{\mathrm{R}} = x_A - x_B$ is governed by the potential U_{K}. Since the reduced mass

$$\frac{M_A M_B}{M_A + M_B} \tag{31}$$

appears as a multiplier associated with x_{R}, the actual motion of x_{R} is Keplerian for the total mass $M_A + M_B$. It should be noted that U_{K} does not come from an existing term in the original Hamiltonian; it is introduced on an *ad hoc* basis to provide the behavior expected by physical arguments. For instance, by collapsing the two subsystems into their respective barycenters one has to obtain the same U_{K}. Starting from the root of the tree, *i.e.*, with the complete system, the relative Keplerian motions of the barycenters accumulate as one "climbs" the branches of the tree.

The potential for the perturbation part between the particles in subsystems A and B can be written as

$$U_{\mathrm{P}} = -\sum_{i \in A,\, j \in B} G m_i m_j \left(\frac{1}{\|x_i - x_j\|} - \frac{1}{\|x_A - x_B\|} \right). \tag{32}$$

This potential is the original sum of the potentials between individual particles in A and B from which U_{K} is subtracted. The perturbation part U_{P} is expressed in terms of the original coordinates x_i and x_j but the intermediate Jacobi coordinates x_A and x_B are also present. The simplest way to deal with U_{P} is to use the original coordinates by noting that the contributions of these to x_A and x_B are known. The kth component of the acceleration on particle i can be computed as

$$\frac{d}{dt} v_{i,k} = -\frac{1}{m_i} \frac{\partial U_{\mathrm{P}}}{\partial x_{i,k}}, \tag{33}$$

i.e.,

$$\frac{d}{dt} v_{i,k} = -\sum_{j \in B} G m_j \left(\frac{1}{\|x_i - x_j\|^3} (x_{i,k} - x_{j,k}) - \frac{1}{\|x_A - x_B\|^3} (x_{A,k} - x_{B,k}) \right). \tag{34}$$

One can store the intermediate Jacobi coordinates x_A and x_B in the tree and retrieve them when needed. In order to reduce roundoff error, one could rewrite the expression of the perturbing accelerations (34) in different forms, *e.g.*, by expanding it in cases when the two terms have nearly equal magnitude. In our experience with integrating the motion of the major planets, however, the roundoff errors introduced in the solution of Kepler's equation (Wisdom and Holman, 1991; Danby, 1992) are more troublesome than those in the computation of the accelerations. When high accuracy is desired, a carefully formulated Cowell-Störmer integrator with special considerations for round-off error reduction (Newman *et al.*, 1990) appears to be preferable.

The construction of the Keplerian and perturbation parts of the potential thus closely parallel each other in the hierarchical approach under consideration. This parallelism makes the actual coding easy, once the binary tree is at hand: One does not need the reduced masses and the Hamiltonian is not required in explicit form; test particles can be inserted anywhere in the tree without the need for any special treatment; and, for the sake of efficiency, one can group together test particles which are at the same location in the tree and process them in a single loop. A straightforward implementation of the scheme, along with other numerical integrators, is provided by Varadi (1996).

5. Application to Hamiltonian perturbation theory

MWSFs offer further advantages for Hamiltonian perturbation computations. In this section we demonstrate how the property of selective cancellation can be exploited. We consider the case of planetary motions but more complicated hierarchical systems can be treated the same way. In the case at hand, the tree of Jacobi coordinates reduces to a chain which starts with the Sun. The planets, having either finite or vanishingly small mass, are added one after the other, in the order of increasing semi-major axes.

Let us consider now Jacobi coordinates and the splitting of the Hamiltonian into Keplerian and perturbations parts. The reduced masses m_i' are

$$m'_i = \beta_i m_i, \tag{35}$$

where

$$\beta_i = \frac{\sum_{j=1}^{i-1} m_j}{\sum_{j=1}^{i} m_j}. \tag{36}$$

The first particle is the Sun, which is associated with the total mass of the system in Jacobi coordinates.

The Keplerian part forms the unperturbed system

$$H_0 = -\sum_{i=2}^{N} \frac{m'_i {\mu_i}^2}{2{L_i}^2}, \tag{37}$$

where

$$\mu_i = G \sum_{j=1}^{i} m_j. \tag{38}$$

The perturbation part can be expanded in the reduced masses as

$$H_{\mathrm{P}} = H_1 + H_2 + \ldots, \tag{39}$$

where H_i is of the $(i+1)$th order in the reduced masses. The first term can be written as

$$H_1 = \sum_{i,j} m'_i m'_j R_{i,j}, \tag{40}$$

where $R_{i,j}$ is a function of the variables of the planets i and j. The β_i factors enter the computations at this step in the definition of $R_{i,j}$; these factors do not vanish even when test particles are involved, since $m_1 \neq 0$.

We consider here the well-known method of Lie-series for computing Hamiltonian perturbations (*e.g.*, Lichtenberg and Lieberman, 1983; Message, 1995), as applied to planetary motions. The calculations involve the evaluation of repeated Poisson brackets and also solving an equation of the form

$$\{H_0, G\} = F, \tag{41}$$

where H_0 denotes the unperturbed Hamiltonian, F is a known function, and G is the term in the Lie-series generator to be computed. Equation (41) is also known as the homological equation (Arnold, 1983). Similar equations appear in various perturbation computations even outside the framework of Hamiltonian dynamics.

The first-order Lie-series generator G_0 can be written as the sum

$$G_0 = \sum_{i,j} m'_i m'_j G_{i,j}, \tag{42}$$

where each term on the right-hand side is the solution of

$$\{H_0, m'_i m'_j G_{i,j}\} = m'_i m'_j R_{i,j}. \tag{43}$$

One can observe that each term in the solution G_0 of the homological equation has the same reduced masses as multipliers as does the corresponding term of the perturbation. The unperturbed Hamiltonian H_0 is linear in the masses, since it is the sum of Keplerian Hamiltonians; thus the reduced masses in the Poisson bracket $\{H_0, G_0\}$ cancel those multiplying H_0 but not the ones multiplying the given term of G_0.

Higher-order computations involve repeated Poisson brackets which could lead to singularities in the masses. This does not happen, at least in meaningful applications, *i.e.*, the reduced masses selectively cancel. This can be easily seen by noting that the Poisson bracket of two arbitrary functions involves nonzero derivatives only with respect to their common variables. If at least one of these functions is proportional to the appropriate mass, then no singularity occurs in the Poisson bracket. If both functions are proportional to the same reduced mass, then their Poisson bracket is proportional to that reduced mass. In the computation of the Lie-series generator terms at orders higher than the first, both the terms of the lower-order Lie-series generator and those of the perturbing Hamiltonian are proportional to the appropriate masses. In the computation of the transformation of coordinates, the Poisson brackets always involve the terms of the Lie-series generator and thus singularities are avoided.

MWSFs have been used to derive a Hamiltonian perturbation theory up to the third order in the masses for the Sun-Jupiter-Saturn system (Varadi *et al.*, 1995b; Ghil *et al.*, 1996). They also helped define the appropriate resonant angle variables for a test particle which is simultaneously affected by the Jupiter-Saturn 2:5 mean-motion resonance and the 1:1 mean-motion resonance with Saturn (Varadi *et al.*, 1995a; De la Barre *et al.*, 1996); such particles were dubbed “Bruins,” by analogy with the Trojans that are in 1:1 resonance with Jupiter.

6. Conclusions

Expressing the standard symplectic form in terms of velocities instead of momenta can simplify various aspects of the gravitational N-body problem. The resulting mass-weighted symplectic forms (MWSFs) help clarify the relationship between the symplectic form and the total kinetic energy. Action-angle variables for the Kepler problem can be formulated in a more natural way using MWSFs; this, in turn, simpli£es some of the symbolic-manipulation aspects of Hamiltonian perturbation theory. The same formalism and the same simpli£cations apply to both finite and vanishingly small masses.

The results presented in this paper can be easily extended to cases not discussed here. For instance, Hamiltonian perturbation theory for a binary asteroid can be formulated by introducing a vanishingly small total mass while keeping the relative masses of the two components finite.

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